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Book review

Thinking About Ordinary Differential Equations by **R. E. O'Malley, Jr.** (Cambridge University Press, 1997, 259 pp.) GB £ 40.00, US $ 69.95 hc, GB £ 14.95, US $ 24.95 pb, ISBN 0 521 55314 8 hc, ISBN 0 521 55742 9 pb.

The author of this recent slim addition to the series **Texts in Applied Mathematics** states in the preface that the book is designed for the advanced undergraduate or beginning graduate student. This text is neither an introduction to, nor a compendium of, the subject of ordinary differential equations and some elementary knowledge of ODE's and matrix algebra is assumed. A careful scrutiny reveals that O'Malley's text is to ODE's what Ian N. Sneddon's book *Elements of Partial Differential Equations* is to PDE's: a concise elegant presentation of essential solution techniques. The fundamental pedagogical difference between the books is the abundance of exercises provided by O'Malley (at the end of each chapter, with explicit solutions for approximately one-third of those posited) who rightfully contends that needed skills are "largely acquired from experience." The titles of the six chapters describe the extent of material covered: First-order equations; Linear second-order equations; Power series solutions and special functions; Systems of linear differential equations; Stability concepts; and Singular perturbation methods. Below I comment on each chapter.

The remarks prefacing Chapter 1 introduce through examples much of the terminology used throughout the text: standard form, order, nonhomogeneous, autonomous, linear and nonlinear, regular and singular, existence and uniqueness, stable and unstable, and systems of equations. Standard first-order equations classified as being separable, exact (or that can be made exact through the art of finding an integrating factor), linear, and of the homogeneous type are introduced in rapid succession, often selecting the same example ODE to show how it may be solved using different analytical methods. Bernoulli and Ricatti equations are conveniently phased into the exposition. The chapter ends with a brief discussion of independent-and dependent-variable absent second-order equations that often lend themselves to closed form solutions by an appropriate change of variables. Appropriate to this introduction, all examples have solutions in closed form, but the reader is warned in the epilog to this chapter that such solutions, explicit or implicit, are rare. At one point in the discussion the author states that "it may take some experience to recognize equations as being of the homogeneous type." This gives the reader the uneasy feeling that discovering such forms is more of an art than a science. In my opinion it would have been better at this juncture to point out that many ODE's governing $y(x)$ can be tested for homogeneity by substituting $x \rightarrow ax$, $y \rightarrow a^m y$ and determining whether a value of m exists for which the original equation is recovered.

Chapter 2 begins with a discussion of linear homogeneous second-order ODE's, showing how to test for linear independence of solutions using the Wronskian and how a second independent solution may be constructed from a known solution by reduction of order. The general procedure for determining the quadratic roots λ characterizing $e^{\lambda x}$ solutions of constant coefficient ODE's is given, leading logically to a discussion of how particular solutions may be obtained in integral form through the use of a Green's kernel function found by factorization. Particular solutions to nonconstant coefficient equations are developed through variation of parameters, resulting in an integral representation using an appropriately modified Green's kernel. Having set forth the formal techniques of factorization and variation of parameters, the speed and power of finding particular

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solutions *via* the method of undetermined coefficients is exhibited. Using annibilators, a table of stratagies for selecting particular solution forms is constructed for constant coefficient equations whose inhomogeneous terms are of polynomial, trigonometric, or exponential type, or any product thereof. The chapter is concluded with example applications to mechanical and electrical oscillator systems, showing resonance phenomena in the former and underdamped, critically damped, and overdamped responses in the latter.

Chapter 3 covers just the amount of material necessary to solve variable coefficient second-order ODE's with regular singular points (Frobenius solutions). In the Taylor series introduction for ordinary point problems, an example is given of a divergent series solution, conveying the notion that such asymptotic power series are of great practical importance. (These little comments throughout the text are clearly meant to peak the student's curiosity, and an appropriate reference is given in each case.) After defining a regular singular point of a differential equation and showing that a Taylor series approach to its solution is guaranteed (in some neighborhood about the point at which initial conditions are prescribed), examples of a regular series solution to an ODE, singular at its initial point or nonlinear, are given. Following the definition of a regular singular point, the Frobenius method is introduced through an example second-order equidimensional equation. Application of the usual solution form $y = x^\lambda$ for these equations (a severly truncated Frobenius series) leads to an indicial equation that exhibits regular solutions for distinct roots λ and $\ln|x|$ behavior for λ with complex or repeated real roots. Thus all the features of Frobenius solutions are captured in this exquisite example without ever invoking the general Frobenius series solution form! What follows is a general discourse on determining the coefficients of a Frobenius expansion, made elegantly short through use of the Cauchy product formula for multiplication of two series. The difficult exceptional solution giving rise to an $\ln|x|$ term when the roots λ differ by an integer is then simply stated, appropriately indulging the reader to accept the result on faith since it was motivated by the simple equidimensional example. Two important applications of the theory serve to define Bessel functions and Legendre polynomials and this is followed by a succinct exposition of Sturm-Liouville boundary-value problems. A final application of the Frobenius method to an equation with an irregular singular point yielding a divergent solution is provided to reinforce the notion of asymptotic approximations, and the chapter ends with many exercises. One typographical error in this chapter is of consequence: in Exercise 18 on page 117 concerning the generating function for Bessel functions J_n, the product $t^n[1 + (-1)^n]$ should be replaced by $[t^n + (-t)^{-n}]$.

Systems of linear ODE's are handled in Chapter 4 using matrices for generality. Eigenvalue and eigenfunction pairs are introduced through a discussion of autonomous, homogeneous intitial-value problems. Examples given for second-order systems readily solved by hand shows how solutions may be written in the form e^{At}, where A is the fundamental matrix. Variation of parameters applied to the general nonhomogeneous equation then leads to the concept of a Green's function in the general convolution solution form involving the exponential matrix e^{At} for A of arbitrary $n \times n$ size. It is then shown how linear inhomogeneous boundary-value problems may be handled with appropriate modification of the Green's function. Next, nonconstant matrices $A(t)$ for variable coefficient equations are considered and solution procedures are developed for cases when $A(t)$ is either a (block) diagonal or a (block) triangular matrix. In the final section of this chapter the matrix exponential for constant matrices is reconsidered to present a clear derivation of its finite spectral decomposition, along with simple examples to exhibit the matrix operations involved in its application.

Chapter 5 is a relatively concise exponsition of stability concepts. Phase space plots of the behavior of linear two-dimensional systems illustrate the various classifications of stable (node, spiral, center) and unstable (saddle) points. The power of phase space analysis for conservative nonlinear systems is exhibited using the classic examples of a simple pendulum and a model combustion problem with Arrhenius kinetics. The stability of higher-dimensional systems is elucidated with the aid of the finite spectral decomposition result for constant coefficient linear equations derived in Chapter 4. This leads to definitions of Liapunov stable, uniformly stable, and asymptotically stable systems, and previous examples given in the chapter are viewed in this light. The chapter concludes with a discussion of Liapunov functions and how they may be used to discern when autonomous systems are stable, asymptotically stable, or unstable. There are a significunt number (I count at least eight) of ommissions of the words "unstable" or "unstable mainifold" in this chapter. These occur both in the fundamental definitions of stability and in discussions of examples, thereby muddling an otherwise clear presentation. A more serious problem lies in Figure 13 which shows a phase plot of angular frequency ω *versus* angular displacement θ for a nonlinear pendulum wherein the separatrices of the equation $\omega(\theta) = \pm\sqrt{2k^2\cos\theta + c}$ intersect as vertical tangents at $\theta = m\pi$ for m odd. This sketch is in error because those points are true saddle points where the separatrices (given by $c = 2k^2$) must intersect with finite slope, readily calculated to be $\pm k$. More unfortunate is the fact that this figure is displayed on the cover of the text.

The final Chapter 6 is a short course in singular perturbation methods and matched asymptotic expansions. Regular and singular perturbation techniques are introduced using (nonlinear) algebraic and (linear) ordinary differential equations, as are the concepts of solution nonuniformity, inner and outer expansions, and composite solutions. The notions are then applied to nonlinear initial and boundary-value problems using some sophisticated, but tractable, examples. The discussion concludes with an asymptotic analysis of a nonlinear ODE, used to model combustion phenomena, which has an exact solution. Here perturbation methods are nicely exploited to show the phenomena of solution bistability.

Much to his credit, O'Malley does not indulge in a discussion of Laplace transform techniques or chaos. Although these topics are relevant in engineering and applied science, there are now many texts devoted to those subjects. As stated in the preface, the goal is to provide practical methods for solving ODE's analytically, with the fundamental perspective that there is no best way to solve a given equation. Most equations encountered by the scientist today are solved numerically, and the author encourages the use of sympbolic programs like *Mathematica, Maple*, or *Matlab* to solve some of the more complicated (nonlinear) examples and, of equal importance, to aid in the visualization of solution behaviors. The reader is reminded, however, that the traditional analytical techniques covered in the text indeed provide the basis for successful computing algorithms. To a significant extent the appeal for this book lies in the wealth of examples and solutions that have been selected more for further understanding than for drill. Through the process of review, I have discovered several mathematical developments that I would like to incorporate in my graduate level fluid mechanics and mathematics courses. I am sure other readers will come away with similar rewards from reading this delightful text.

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